CALCULUS.

269. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Prove that $\int_0^1 (x^a+x^{-a})\log\left(\frac{1+x}{1-x}\right)\frac{dx}{x}=\frac{\pi}{a}\tan(\frac{1}{2}\pi a)$.

Solution by S. A. COREY, Hiteman, Iowa.

Developing $(x^a+x^{-a})\log\left(\frac{1+x}{1-x}\right)\frac{1}{x}$ into a power series in x and integrating, we have

$$\int (x^a+x^{-a})\log\left(\frac{1+x}{1-x}\right)\frac{dx}{x}=2\left[\frac{x^{1+a}}{1+a}+\frac{x^{1-a}}{1-a}+\frac{x^{3+a}}{3(3+a)}+\frac{x^{3-a}}{3(3-a)}+\dots\right]$$

and the definite integral of the problem becomes equal to the series,

$$2\left[\frac{1}{1+a}+\frac{1}{1-a}+\frac{1}{3}\cdot\frac{1}{3+a}+\frac{1}{3}\cdot\frac{1}{3-a}+\dots\right]$$

which reduces to the form,

$$4\left[\frac{1}{1^2-a^2}+\frac{1}{3^2-a^2}+\frac{1}{5^2-a^2}+\dots\right]. \tag{1}$$

Fourier's cosine series for $\cos(ax)$ is

$$\cos(ax)=\frac{2a\sin(a\pi)}{\pi}\left[\frac{1}{2a^2}+\frac{\cos x}{1^2-a^2}-\frac{\cos 2x}{2^2-a^2}+\frac{\cos 3x}{3^2-a^2}-\dots\right], \tag{2}$$

where a is fractional. When $x=\pi$, (2) gives

$$\cos(a\pi)=\frac{2a\sin(a\pi)}{\pi}\left[\frac{1}{2a^2}-\frac{1}{1^2-a^2}-\frac{1}{2^2-a^2}-\frac{1}{3^2-a^2}-\dots\right], \tag{3}$$

and when $x=0$, it gives

$$1=\frac{2a\sin(a\pi)}{\pi}\left[\frac{1}{2a^2}+\frac{1}{1^2-a^2}-\frac{1}{2^2-a^2}+\frac{1}{3^2-a^2}-\dots\right]. \tag{4}$$

Subtracting (3) from (4), we have

$$1-\cos(a\,\pi)=\frac{4a\sin(a\,\pi)}{\pi}\left[\frac{1}{1^2-a^2}+\frac{1}{3^2-a^2}+\frac{1}{5^2-a^2}+\ldots.\right], \tag{5}$$

whence the value of the series, (1), when $0<a>1$, is readily found to be $\frac{\pi}{a}\tan\left(\frac{\pi a}{2}\right)$, as required by the problem.

When $a>1$, or $a<-1$, let $b=a^{-1}$, and substitute in the problem. The definite integral does not change its form by making this substitution, but the right hand member of the equation does change. Hence the given equation does not hold for values of $a>1$, or <-1. It can readily be shown that the equation holds when $a=0$, $a=\pm1$, and when $0>a>-1$. The range of values of a in which the given equation holds is, then, $-1 \overline{\gtrless} a \overline{\gtrless} 1$.

Also solved by C. N. Schmall, V. M. Spunar, J. Scheffer, and the Proposer.

270. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\sum\limits_{x=0}^{x=\infty}\frac{1}{(a^2+x^2)^n}=\frac{\pi}{2a^{2n-1}}\cdot\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{(2n-3)}{(2n-2)}\cdot\frac{1}{2a^{2n}}$, n being a positive integer >1.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; C. N. SCHMALL, New York City; and J. SCHEFFER, A. M., Hagerstown, Md.

Assuming $\sum\limits_{x=0}^{x=\infty}\frac{1}{(a^2+x^2)^n}\equiv\int_0^\infty\frac{dx}{(a^2+x^2)^n}$, the integral may be evaluated in two different ways.

(1) Assume the equation

$$\int_0^\infty\frac{dx}{x^2+\alpha}=\frac{\pi}{2}\cdot\frac{1}{\alpha^{\frac{1}{2}}}.$$

Now differentiate both sides $(n-1)$ times with regard to α, we obtain

$$\int_0^\infty\frac{dx}{(x^2+\alpha)^n}=\frac{\pi}{2}\,\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}\,\frac{1}{\alpha^{n-\frac{1}{2}}};$$

and substituting a^2 for α, we have

$$\int_0^\infty\frac{dx}{(a^2+x^2)^n}=\frac{\pi}{2}\cdot\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}\cdot\frac{1}{a^{2n-1}}.$$

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(2) Or, by putting $x=a\tan\theta$ in the given integral, we have

$$\int_0^\infty\frac{dx}{(a^2+x^2)^n}=\frac{1}{a^{2n-1}}\int_0^{\frac{1}{2}\pi}(\sin\theta)^{2n-2}d\,\theta=\frac{1}{a^{2n-1}}\cdot\frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)}\cdot\frac{\pi}{2}.$$

SCHMALL, ZERR, SCHEFFER.